Dynamics of a bouncing matter-wave soliton under periodically modulated atomic interactions

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Аналитически и численно рассмотрена динамика солитонов волн материи, отскакивающих от отражающей поверхности (атомного зеркала) под действием силы тяжести, когда двухчастичные взаимодействия между атомами периодически модулируются во времени. Аналитическое описание основано на вариационном подходе. Получены частоты малых колебаний центра массы и ширины солитона. Аналитические предсказания сопоставлены с результатами численного моделирования уравнения Гросса-Питаевского и найдено качественное согласие между ними.

Ключевые слова: конденсат Бозе-Эйнштейна, поле гравитации, вариационное приближение, частота колебаний.

We consider, by analytical and numerical means, the dynamics of a matter-wave soliton bouncing on the reflecting surface (atomic mirror) under the influence of gravity, when two-body interactions between atoms are periodically modulated in time. The analytical description is based on the variational approach. The frequencies of small amplitude oscillations of the soliton's center-of-mass and width are obtained. Analytical predictions are compared to the results of numerical simulations of the Gross-Pitaevskii equation and qualitative agreement between them is found.

Keywords: Bose-Einstein condensate, field of gravity, variational approach, oscillation frequencies.

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I. Introduction

The problem of a ball, bouncing under influence of gravitational field, on the oscillating mirror (the Fermi-Ulam model) is one of the basic models of the theory of dynamical systems, which is used for the investigation of such phenomena as Fermi acceleration [1] and dynamical chaos [2-4].

The analogue of this problem in Bose-Einstein condensate (BEC) is the bouncing of a matter-wave packet in the gravitational field above the oscillating wall (atomic mirror created by a laser beam). Altering the position of the mirror in time can emulate the Fermi-Ulam problem for matter-wave solitons, paving the way to observe many interesting phenomena, such as Fermi acceleration for cold atoms and solitons, interference of matter-wave packets, nonlinear resonances, hysteresis and dynamical chaos with matter-wave solitons [5-8]. This model is also important for the realization of time crystals [9].

It is interesting to study the periodically driven interactions between atoms and their influence on the dynamics of bouncing solitons. While the translational symmetry is not breaking by the temporal modulations of interactions, we can expect the influence of the modulations on the motion of the soliton. This possibility appears since the effective potential acting on the center-of-mass of soliton due to interaction with the mirror depends on the width of the soliton. Under the modulated interactions the width is oscillating and so we can expect the oscillations of the

effective potential. The variations of the strength of two-body interactions via the atomic scattering length can be achieved using the Feshbach resonance management technics [10].

Following the above mentioned idea, we will study the bouncing of the matter-wave soliton on the standing mirror when the atomic scattering length is periodically modulated in time. We will explore the possibility in accelerating of the soliton under such a perturbation. This model can be also considered as a prototype for the nonlinear time crystal with managed nonlinearity. The Feshbach resonance method [11-12] allows changing the force of interaction between atoms in a condensate, using an external magnetic or light field. This method has opened up great opportunities for studying the properties of the condensate.

II. Model

The model is based on the following 1D Gross-Pitaevskii equation (GPE):

$$i\hbar\frac{\partial\psi}{\partial t}=-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2}+mgx\psi+U(x)\psi+2\hbar\omega_\perp a_s|\psi|^2\psi\ , \tag{1}$$

where $\psi(x,t)$ is the wave function of the condensate trapped in a tight quasi-1D trap, x is the spatial coordinate of the wave packet above the horizontal atomic mirror, represented by the reflecting potential $U(x)$, g is the strength of the gravitational potential, $\omega_\perp$ is the trap frequency in the tightly confining radial direction, and m, a_s are the atomic mass and s-wave scattering length, respectively. The gravitational units of space and time are defined as follows

$$l_g=(\hbar^2/m^2g)^{1/3},\qquad t_g=(\hbar/mg^2)^{1/3}, \tag{2}$$

which enable to rewrite the Eq. (1) in dimensionless form

$$i\psi_t+\frac{1}{2}\psi_{xx}+\alpha x\psi+U(x)\psi+\gamma(t)|\psi|^2\psi=0, \tag{3}$$

where the new variables are defined as $x\to x/l_g$, $t\to t/t_g$, $U(x)=U(x)/mgl_g$, $\psi\to\sqrt{2\omega_\perp|a_s|t_g}\,\psi$.

To simplify further calculations we use the expression $U(x) = V_0\delta(x)$ for the reflecting barrier (atomic mirror). Hence, the external potential becomes the combination of a linear potential and delta-function type reflecting barrier

$$V(x)=\alpha x+V_0\delta(x), \tag{4}$$

where α is a parameter that represents the strength of the field of gravity, V_0 is the strength of delta-potential.

The time variation of the s-wave scattering length a_s has the following form [10]:

$$a_s=a_{bg}\left(1-\frac{\Delta}{B(t)-B_0}\right), \tag{5}$$

where a_{bg} is the scattering length outside the resonance (background scattering length), B_0 is magnetic field at which resonance occurs, Δ is the width of resonance at $B(t) = B_0$.

By taking Eq.(5) into account, the time dependent nonlinearity coefficient can be written as follows

$$\gamma(t)=\gamma_0+\epsilon\sin(\Omega t), \tag{6}$$

where $\gamma_0=2\hbar\omega_\perp a_s$ is the unperturbed coefficient of nonlinearity, ϵ, Ω are the amplitude and frequency of modulation of nonlinearity.

III. Variational approximation

It is easy to verify that Eq. (3) can be retrieved from the following Lagrangian density by utilizing the Euler-Lagrange equations

$$L=\frac{i}{2}(\psi\psi_t^*-\psi^*\psi_t)+\frac{1}{2}|\psi_x|^2-V(x)|\psi|^2-\frac{\gamma(t)}{2}|\psi|^4. \tag{7}$$

We will look for the solution in the form of hyperbolic secant function, which is the bright soliton solution of GPE in absence of the external potential $V(x)$

$$\psi(x,t)=A\,\mathrm{sech}\left(\frac{x-\zeta}{a}\right)e^{ib(x-\zeta)^2+iv(x-\zeta)+i\varphi}, \tag{8}$$

where $A(t)$, $a(t)$, $\zeta(t)$, $v(t)$, $b(t)$, $\varphi(t)$ − are the time-dependent variational parameters, meaning the amplitude, width, position of the center-of-mass, velocity, chirp and phase of the soliton, respectively. The norm of the trial function (8) represents the conserved quantity of the governing Eq. (3)

$$N=\int_{-\infty}^{\infty}|\psi|^2\,dx=2aA^2. \tag{9}$$

Substituting the trial function (8) into Eq. (7) and integrating over the space variable $\bar{L}=\int_{-\infty}^{\infty}L dx$, we get the averaged Lagrangian

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$$\bar{L} = N\left[\frac{\pi^2}{12}a^2 b_t + \frac{\pi^2}{6}a^2 b^2 - \frac{1}{2}\zeta_t^2 + \varphi_t + \frac{1}{6a^2} - \right.$$
$$\left. - \alpha\zeta - \frac{V_0}{2a}\operatorname{sech}^2\left(\frac{\zeta}{a}\right) - \frac{\gamma(t)N}{6a}\right]. \quad (10)$$

Applying the Euler-Lagrange equations to Eq. (10) leads to the following system of equations for the width and center-of-mass position of the soliton with a time-dependent coefficient of nonlinearity

$$a_{tt} = \frac{4}{\pi^2 a^3} - \frac{6V_0}{\pi^2 a^2}\operatorname{sech}^2\left(\frac{\zeta}{a}\right)\left[1 - \frac{2\zeta}{a}\tanh\left(\frac{\zeta}{a}\right)\right] - \frac{2\gamma(t)N}{\pi^2 a^2}, \quad (11)$$

$$\zeta_{tt} = \alpha - \frac{V_0}{a^2}\operatorname{sech}^2\left(\frac{\zeta}{a}\right)\tanh\left(\frac{\zeta}{a}\right). \quad (12)$$

These equations can be written in the form of Newton's equations $a_{tt} = -\partial U_1 / \partial a$, $\zeta_{tt} = -\partial U_2 / \partial \zeta$ for a unit mass particle in an anharmonic potential

$$U_1(a) = \frac{2}{\pi^2 a^2} - \frac{6V_0}{\pi^2 a}\operatorname{sech}^2\left(\frac{\zeta}{a}\right) - \frac{2\gamma(t)N}{\pi^2 a}, \quad (13)$$

$$U_2(\zeta) = -\alpha\zeta - \frac{V_0}{2a}\operatorname{sech}^2\left(\frac{\zeta}{a}\right). \quad (14)$$

Expressions for potential functions Eq. (13) and Eq. (14) are obtained assuming one of the variables (ζ or a) is constant, corresponding to the minimum of the potential, while the other changes slightly near its equilibrium. More general expression for the potential can be obtained as follows. We rewrite the Lagrange function with kinetic and potential energy parts separated, excluding the chirp and phase parameters:

$$\bar{L} = T - U = \frac{a_t^2}{2} + \frac{\beta\zeta_t^2}{2} - \frac{2}{\pi^2 a^2} + \frac{12}{\pi^2}\alpha\zeta + \frac{6V_0}{\pi^2 a}\operatorname{sech}^2\left(\frac{\zeta}{a}\right) + \frac{2\gamma(t)N}{\pi^2 a}, \quad (15)$$

where $\beta = 12/\pi^2$, and the expression for general potential function takes the following form:

$$U(\zeta, a) = \frac{2}{\pi^2 a^2} - \frac{12}{\pi^2}\alpha\zeta - \frac{6V_0}{\pi^2 a}\operatorname{sech}^2\left(\frac{\zeta}{a}\right) - \frac{2\gamma(t)N}{\pi^2 a}. \quad (16)$$

Now we perform the following numerical experiment. We choose the equilibrium point above the reflecting surface as the initial position of the soliton's center-of-mass, where the force of attraction due to gravity is balanced by the force of repulsion of the soliton from the surface (delta-barrier). This corresponds to the minimum point on the potential curve $U_2(\zeta)$, $\zeta = 2.77$. In this case, the width of the soliton is determined from the condition of the minimum on the curve $U_1(a)$ and is equal to $a_0 = 0.89$ (see Fig. 1a, b). The fixed points (ζ_0, a_0) can be found using the Eq. (16) as well by searching the local minimum in two-dimensional parameter space. Apparently the minimum is located within the dashed island of Fig. 1c

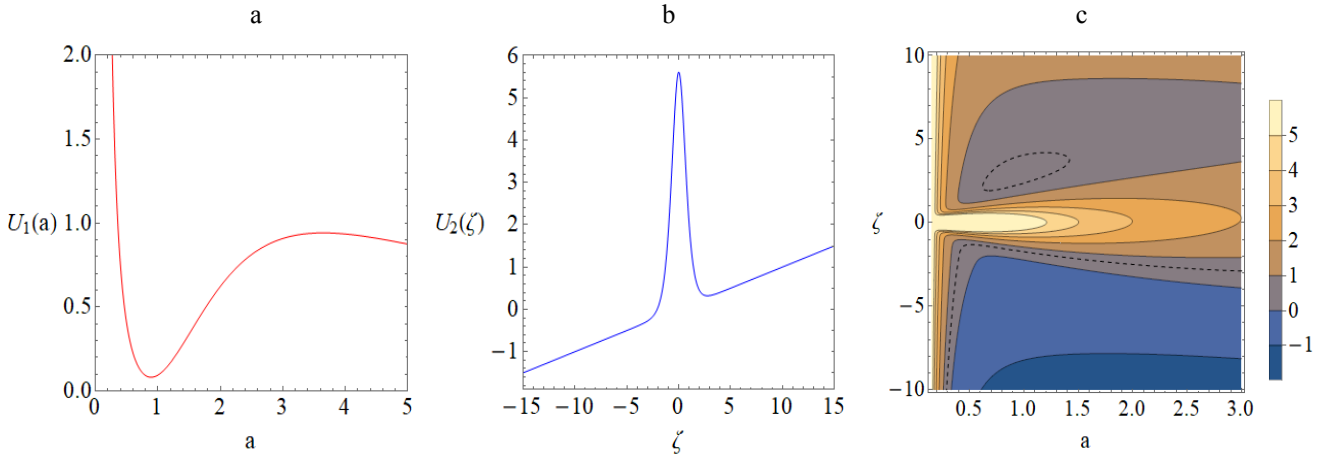


Figure 1. Potential function for width (a) and center-of-mass position (b) of the soliton. A parametric plot for general potential function (c). Parameter values are $\alpha = -0.1$, $V_0 = -10$, $N = 1$, $\gamma_0 = 1$, $\epsilon = 0$, $\zeta_0 = 2.77$, $a_0 = 0.89$.

IV. Normal modes

To analyze small amplitude dynamics of the system we linearize the coupled ordinary differential equations (11)-(12), derived from the variational approximation method (VA). To this end we expand the corresponding solutions near the equilibrium points $a = a_0 + \delta a$, $\zeta = \zeta_0 + \delta\zeta$, where δa and $\delta\zeta$ are small displacements:

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$$\delta a_{tt} + \omega_a^2 \delta a = C_1 \delta\zeta + \Lambda \sin(\Omega t), \quad (17)$$

$$\delta\zeta_{tt} + \omega_\zeta^2 \delta\zeta = C_2 \delta a, \quad (18)$$

where ω_a and ω_ζ are called partial frequencies [13], which can be derived from original equations at stationary points, C_1, C_2, Λ are constants that depend on the initial parameters

$$\omega_a^2 = \frac{12}{\pi^2 a_0^4} - \frac{4\gamma(t)N}{\pi^2 a_0^3} - \frac{12V_0}{\pi^2 a_0^3}\operatorname{sech}^2\left(\frac{\zeta_0}{a_0}\right)\times \times\left[1 - \frac{\zeta_0^2}{a_0^2} - 4\frac{\zeta_0}{a_0}\tanh\left(\frac{\zeta_0}{a_0}\right) + 3\frac{\zeta_0^2}{a_0^2}\tanh^2\left(\frac{\zeta_0}{a_0}\right)\right], \quad (19)$$

$$\omega_\zeta^2 = \frac{V_0}{a_0^3}\operatorname{sech}^2\left(\frac{\zeta_0}{a_0}\right)\left[1 - 3\tanh^2\left(\frac{\zeta_0}{a_0}\right)\right], \quad (20)$$

$$C_1 = \frac{12V_0\zeta_0}{\pi^2 a_0^4}\operatorname{sech}^2\left(\frac{\zeta_0}{a_0}\right)\left[1 + \frac{2a_0}{\zeta_0}\tanh\left(\frac{\zeta_0}{a_0}\right) - 3\tanh^2\left(\frac{\zeta_0}{a_0}\right)\right], \quad (21)$$

$$C_2 = \frac{V_0\zeta_0}{a_0^4}\operatorname{sech}^2\left(\frac{\zeta_0}{a_0}\right)\left[1 + \frac{2a_0}{\zeta_0}\tanh\left(\frac{\zeta_0}{a_0}\right) - 3\tanh^2\left(\frac{\zeta_0}{a_0}\right)\right], \quad (22)$$

$$\Lambda = -\frac{2\epsilon N}{\pi^2 a_0^2}. \quad (23)$$

At $\epsilon = 0$ the time-dependent coefficient of nonlinearity becomes constant, $\gamma(t) = \gamma_0$. Thus, it is convenient to seek for the solutions of Eq. (17) and Eq.(18) as $\delta a = pe^{i\Omega t}$, $\delta\zeta = qe^{i\Omega t}$, where p and q are real parameters representing the amplitude of small oscillations for width and center of mass position, respectively. After performing the corresponding derivatives and inserting the outcome into the equations, the following 4-th order algebraic equation for the frequency Ω is obtained

$$\Omega^4 - (\omega_a^2 + \omega_\zeta^2)\Omega^2 + \omega_a^2\omega_\zeta^2 - C_1C_2 = 0. \quad (24)$$

Solution of Eq. (24) gives us the frequencies, which are known as *normal modes*

$$\Omega_{1,2}^2 = \frac{\omega_a^2 + \omega_\zeta^2 \mp \sqrt{(\omega_a^2 - \omega_\zeta^2)^2 + 4C_1C_2}}{2}, \quad (25)$$

It is evident from the Eq. (25) that the normal frequencies lie outside the interval of partial frequencies, that is, $\Omega_1 < \omega_\zeta$ and $\Omega_2 > \omega_a$ (to be specific, we consider $\omega_\zeta < \omega_a$). Relations of this kind for systems with many degrees of freedom are known under the name of Rayleigh theorems [14].

Dependence of $\Omega_{1,2}$ on ω_ζ is shown in Figure 2a. It can be seen that the difference between the normal frequencies $\Omega_{1,2}$ and the partial $\omega_{a,\zeta}$ at small $\sqrt{C_1C_2}$ is insignificant everywhere, except for the region of degeneracy $|\omega_a^2 - \omega_\zeta^2| \lesssim \sqrt{C_1C_2}$. At sufficiently small ω_ζ, one of the normal frequencies becomes imaginary, the system ceases to be stable.

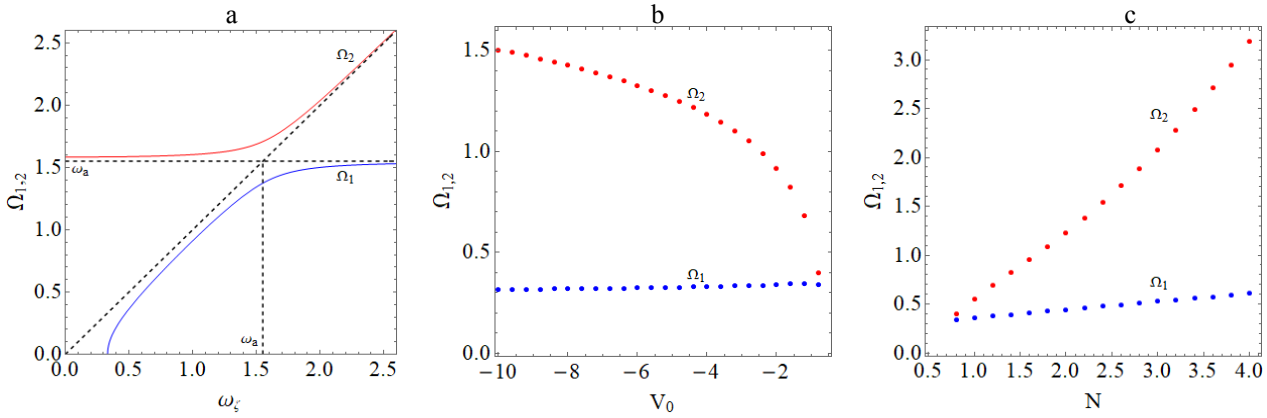


Figure 2. a) Relation between the normal and partial frequencies. Parameters: $V_0 = -10$, $N = 1$, $\zeta_0 = 2.77$, $a_0 = 0.89$, b) and c) describe the dependence of normal frequencies on the strength of reflecting barrier and the norm of matter wave, when $N = 0.8$, and $V_0 = -0.8$, respectively. Other parameters $\alpha = -0.1$, $\gamma_0 = 1$ and $\epsilon = 0$ are the same for all three figures.

The Figure 2b illustrates that Ω_2 declines faster with respect to V_0, when the norm is fixed, while Ω_1 holds its initial value. On the contrary, if the strength of reflecting barrier is constant and the norm of the

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soliton increases, then Ω_2 grows faster than Ω_1.

Moreover at certain values of initial parameters (see Fig. 2b,c), the normal frequencies become almost equal $\Omega_1 \approx \Omega_2$.

V. Numerical analysis

Numerical simulations of Eq. (17) and Eq. (18) give the opportunity to observe if resonance phenomena in the system take place. With this objective we periodically change the coefficient of nonlinearity in time according to Eq. (6) with frequencies $\Omega_{1,2}$, estimated using Eq. (25). The result is presented in Fig. 3.

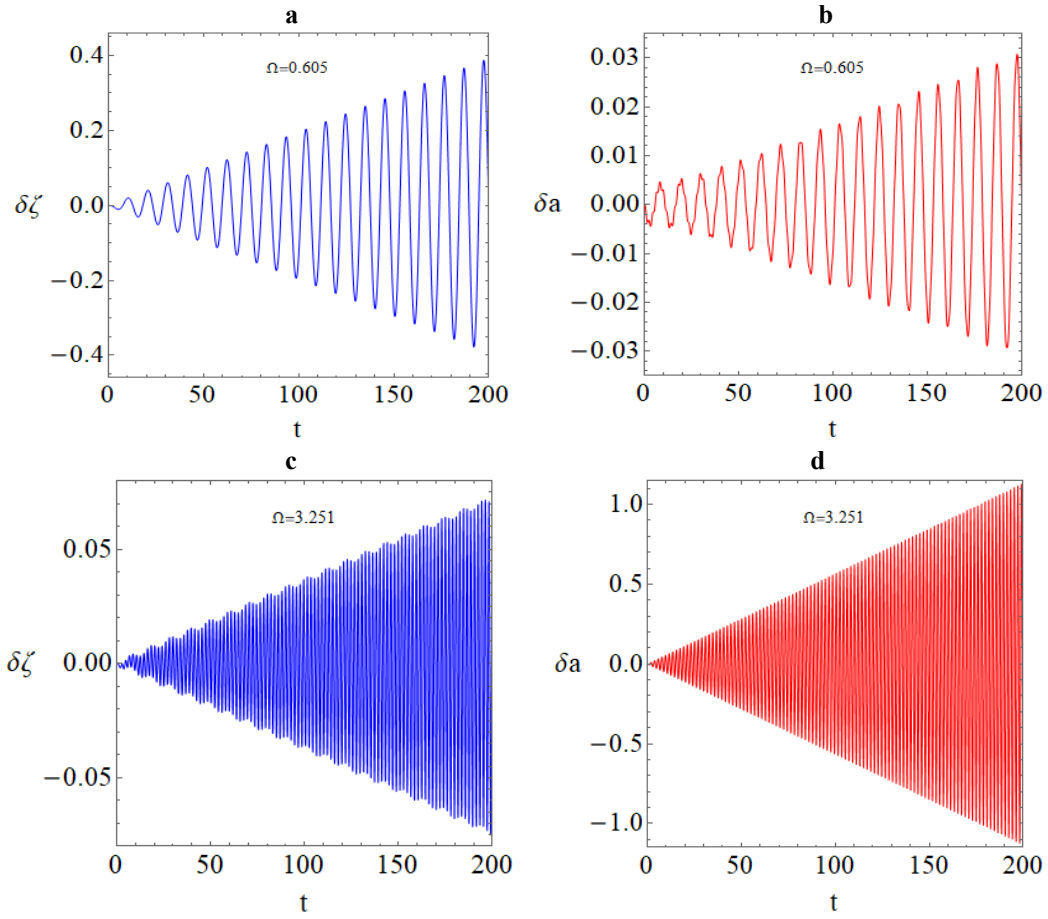


Figure 3. Small oscillation dynamics of the center of mass and width Eqs. (17)-(18) of a soliton with a periodic variation of the coefficient of nonlinearity. Parameter values: $\alpha = -0.1$, $V_0 = -1$, $\gamma_0 = 1$, $N = 4$, $\zeta_0 = 1.21$, $a_0 = 0.46$, $\epsilon = 0.01$, $\Omega_1 = 0.605$, $\Omega_2 = 3.251$.

As it was expected from coupled equations (17)-(18), which include the term $\Lambda\sin(\Omega t)$ that acts on the system as external force, the resonance occurs at both of the normal frequencies

Finally, we compare the prediction of the variational approximation for the position of the center-of-mass Eq. (11) and width Eq. (12) of the soliton and numerical solution of the Gross-Pitaevskii equation (see Fig. 4).The initial position of the center-of-mass corresponds to a stationary position above the surface ζ_0, for which the downward gravitational force is balanced by the upward repulsion from the surface. The modulation frequency of the nonlinearity is equal to the frequency of small amplitude oscillations of the center-of-mass and the width of the soliton near the equilibrium points. This corresponds to the condition of nonlinear resonance. It can be noticed that when perturbation coefficient ϵ is sufficiently small, VA method gives quite accurate predictions, consistent with numerical simulations of the GPE (see Fig. 4a,b). However, if ϵ gets bigger values, the results deviate for the width parameter (Fig. 4.c-f).

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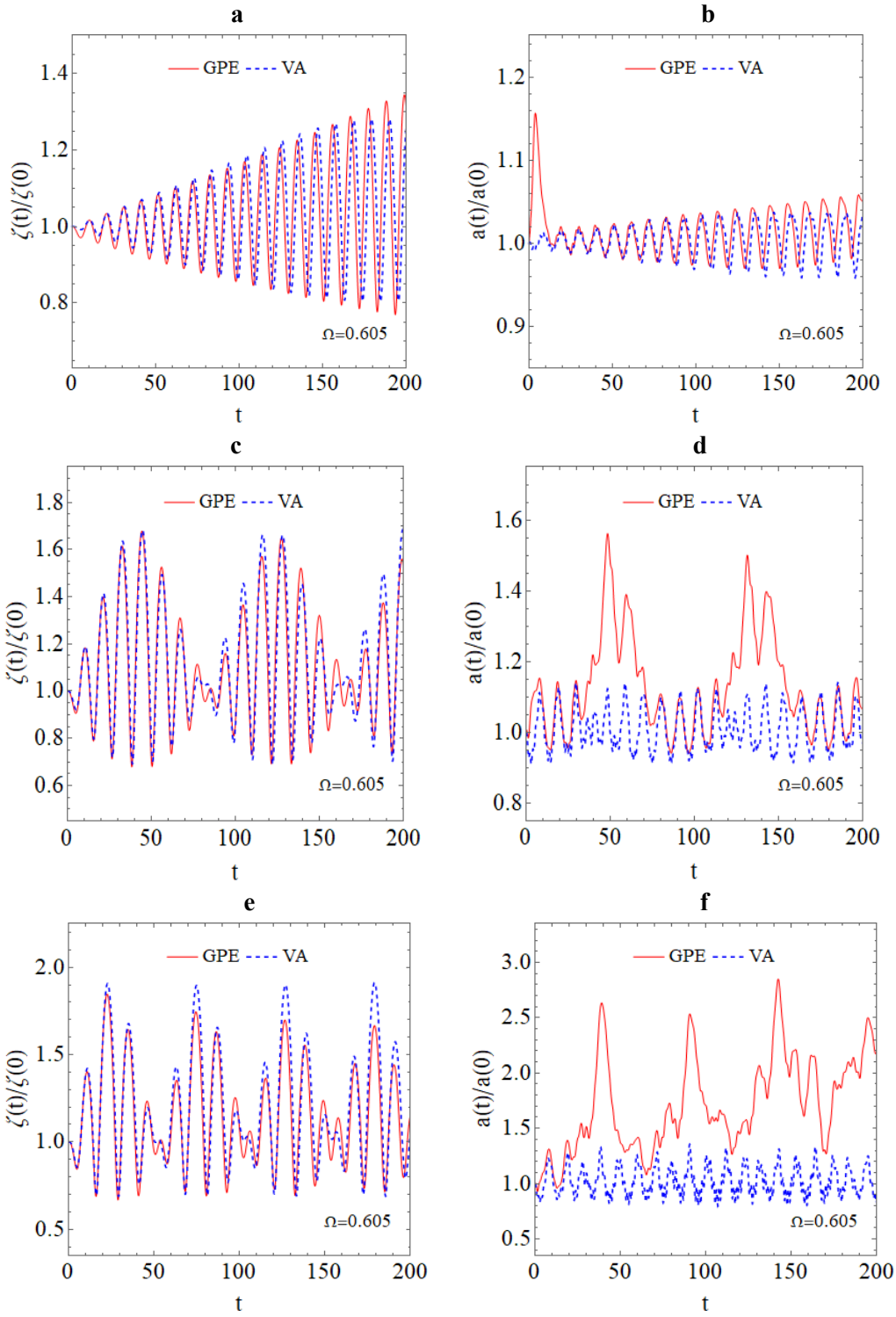


Figure 4. Comparison of the time-dependence of center-of-mass and width of the soliton, when the coefficient of nonlinearity is periodically varied in time. Parameter values: $\alpha = -0.1$, $V_0 = -1$, $N = 4$, $\zeta_0 = 1.21$, $a_0 = 0.46$, $\gamma_0 = 1$, $\Omega_1 = 0.605$. The perturbation coefficient $\epsilon = 0.01$ for a-b, $\epsilon = 0.1$ for c-d, $\epsilon = 0.2$ for e-f. For visual convenience we plot $\zeta(t)/\zeta_0$ and $a(t)/a_0$.

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VI. Conclusion

We have studied the bouncing dynamics of a matter-wave soliton above the reflecting potential barrier (atomic mirror), under the influence of gravitational field and periodically varying in time atomic interactions. To investigate the dynamics of soliton's width and center-of-mass coordinate the variational approach has been employed, and a system of coupled nonlinear ODEs for these parameters is derived.By analyzing the small amplitude dynamics of the soliton we find the normal mode frequencies, that are consistent with numerical simulations of the governing GPE. The nonlinear resonances in the matter-wave soliton's width and center-of-mass position are observed.

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Даврий модуляцияланган атом ўзаро таъсири остида сакровчи материя-тўлқин солитонининг динамикаси

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Атомлар орасидаги икки жисмли ўзаро таъсирлашувлар вақт бўйича даврий модуляцияланганда, тортишиш майдони таъсирида қайтарувчи юзада (атом кўзгу) материя-тўлқини солитонининг сакраш динамикаси аналитик ва сонли усуллар ёрдамида кўриб чиқилади. Аналитик тавсиф вариацион яқинлашув методига асосланади. Солитон масса маркази ва кенглигининг кичик амплитудали тебранишлари частоталари олинади. Аналитик усулда олинган натижалар Гросс-Питаевски тенгламасининг сонли ечими билан таққосланади ва улар ўртасидаги сифат мувофиқлиги топилади.

Калит сўзлар: Бозе-Эйнштейн конденсати, тортишиш майдони, вариацион яқинлашув, тебраниш частоталари.

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