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Spinless particle motion near black hole horizon

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Проанализировано движение бесспиновой частицы в окрестности тяжёлого центрального тела. Выведены волновая функция и энергетический спектр частицы. Показаны основное и первое возбуждённое состояния частицы. Рассматриваемая задача связана с полуклассической формулировкой гравитации, где материя квантуется, а гравитация нет.

Ключевые слова: уравнение Шрёдингера, метрика Шварцшильда, потенциальное поле Шварцшильда, энергетические уровни..

The motion of a light spinless particle in the vicinity of a heavy central mass is analyzed. We have derived the wave function and energy spectrum ofa particle. It is shown the ground state of a particle in the centrally symmetric gravitational field. The excited state of a particle is considered. The given problem relates to a semi-classical formulation of gravity, where matter is quantized, but gravity is not. One of the aims of this study to show the quantum behavior of a particle near to a black hole.

Keywords: Schrödinger equation, Schwarzschild metric, Schwarzschildgravitational potential, gravity, energy levels

I. Introduction

A problem of gravitational radiation of a particle falling in a Schwarzschild black hole has been investigated in late 60-s and early 70-s of the last century [1-9]. The problem of a radial fall is of considerable interest [10] and has been studied by different authors [11, 12]. One of the first studies of a problem of radial falling and radiation was investigated by Ruffini and Wheeler [7]. They had calculated of gravitational radiation of a particle using linearized theory of gravity. Davis et al. [8] had estimated a total gravitational energy radiated by a particle. According to [8] this estimate was 6 times larger than Zerilli's [4] and 4 times larger than Ruffini and Wheeler's [7]. Thorne [3] had studied radiation of gravitational waves from pulsating neutron stars. Detailed calculations of gravitational radiation from slowly rotating fully relativistic stars were done by Ipser [5]. The gravitational waves from a collapse problem were considered by Israel et al [6].

One of the significant studies of falling particle in a centrally symmetric gravitational field of a massive body was extensively studied by different authors [1, 13, 14]. The author of work [13] had investigated a classical case of falling particle with small mass in a gravitational field of a heavy massive body. Further, according to the general theory of relativity, the relation between mass of a particle and the centrally symmetric gravitational field in a point of a particle has been derived. Further, Voracek had deduced the relativistic gravitational potential. And he also has found the relationship of potential and the rest mass of a particle.

It is necessary to note, that the gravitational analogue of a problem of hydrogen atom, where a particle of negligible mass is gravitationally bound to a heavy massive body, can also be considered theoretically. One of the aims of given study, is to consider how given problem can be related to the general relativity, where the gravitational interactionis determined by a Schwarzschild field.

Dirac had considered a classical problem of radial falling particle in the central body [15]. In this paper, we would like to consider a quantum problem of a particle falling into the central body.

The paper is organized as follows. In Section II, we briefly describe the model and governing equation. In particular, in the linear approximation the motion equation for a particle is derived. Section III is devoted to a weak gravitational field. In particular, an expression for quasi-energy levels is obtained in the linear approximation. In concluding, Section IV, we summarize our findings.

We use the metric with signature $(+---)$. Latin indices are taken on the values 1, 2, 3.

II. Schwarzschild source in a gravitational field

We start our consideration from the Schwarzschild metric in the form [14]:

$$ds^2 = \left(1-\frac{r_g}{r}\right)c^2dt^2 - \frac{dr^2}{1-\frac{r_g}{r}} - r^2(d\theta^2+\sin^2\theta d\varphi^2), \quad (1)$$

where r_g=2GM/c^2 is a gravitational radius, G is a gravitational constant, M is a total mass of the bodies producing the field.

We shall consider free falling particle in a gravitational field of a heavy massive body with the mass incomparably greater than the mass of a particle. Below we consider the radial falling, i.e. $d\theta/dt$=$d\varphi/dt$=0. The source of a gravitational field of a massive body is described by the relativistic gravitational potential energy given by [13]:

$$U(r) = mc^2\left(\sqrt{1-\frac{r_g}{r}}-1\right), \quad (2)$$

m is a particle mass.

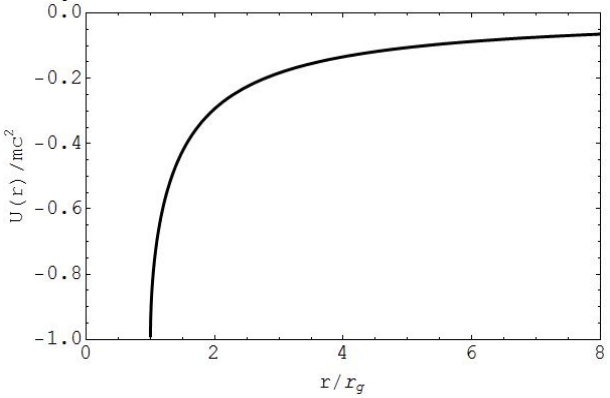


Figure 1. The shape of the relativistic potential energy as a function of distance.

The general form of the dependence of relativistic potential energy $U(r)$ on the distance is illustrated in Fig. 1.

Let us to consider, by means of a simple quantum mechanical model based on the Schrödinger equation (SE), falling of a particle in a gravitational field of a heavy body. The time-independent SE in curvilinear coordinates is of the form [16, 17]:

$$g^{ij}\left(\frac{\partial^2\psi}{\partial x^i\partial x^j}-\Gamma_{ij}^k\frac{\partial\psi}{\partial x^k}\right)+\frac{2m}{\hbar^2}[E-U(r)]\psi=0, \quad (3)$$

where g^{ij} is the metric tensor. The symbols Γ_{ij}^k are denoted Christoffel symbols. In particular, for $\Gamma_{11}^1 = -r_g/2r^2(1-r_g/r)$.

Equation (3) for the radial function has the form

$$\left(1-\frac{r_g}{r}\right)\frac{d^2\varphi}{dr^2}+\frac{1}{2r}\left(4-\frac{3r_g}{r}\right)\frac{d\varphi}{dr}+ \\ +\frac{2m}{\hbar^2}\left[E-mc^2\left(\sqrt{1-\frac{r_g}{r}}-1\right)\right]\varphi=0. \quad (4)$$

We have written the symbol of the total derivative with respect to r because the φ-function depends only on r.

Let us consider falling of a particle in a vicinity of the central body, so $r = r_g+x$ with x small, and let us neglect x^2. We interested in the case $r_g/r \sim 1$. Then Eq. (4) becomes

$$\frac{x}{r_g}\frac{d^2\varphi}{dx^2}+\frac{1}{2r_g}\left(1+2\frac{x}{r_g}\right)\frac{d\varphi}{dx}+\frac{2}{\lambda^2}\left[\frac{E}{mc^2}+1-\sqrt{\frac{x}{r_g}}\right]\varphi=0, \quad (5)$$

where $\lambda = \hbar/mc$ is the constant.

Here it is convenient to introduce a dimensionless variable $y = x/r_g$. Then we have the dimensionless equation

$$y\frac{d^2\varphi}{dy^2}+\left(y+\frac{1}{2}\right)\frac{d\varphi}{dy}+2\left(\frac{r_g}{\lambda}\right)^2\left[\frac{E}{mc^2}+1-\sqrt{y}\right]\varphi=0. \quad (6)$$

Eq. (6) may be solved by replacement [18]:

$$\varphi(y)=\eta(\xi),\ \ y=\xi^2,\ \ \xi_y=\frac{1}{2\xi},\ \ \xi_{yy}=\frac{1}{4\xi^3}, \\ \varphi_y=\frac{1}{2\xi}\frac{\partial\eta}{\partial\xi},\ \ \varphi_{yy}=\frac{1}{4\xi^2}\frac{\partial^2\eta}{\partial\xi^2}-\frac{1}{4\xi^3}\frac{\partial\eta}{\partial\xi}. \quad (7)$$

Thus Eq. (6) may be written in the form

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$$\frac{d^2\eta}{d\xi^2}+2\xi\frac{d\eta}{d\xi}+8\left(\frac{r_g}{\lambda}\right)^2\left[\frac{E}{mc^2}+1-\xi\right]\eta=0. \qquad (8)$$

The solution of Eq. (8) has the asymptotic behavior

$$\eta(\xi)\sim\exp\left[-\xi^2-\left(\frac{2r_g}{\lambda}\right)^2\xi\right]H_n\left(\xi+\left(\frac{2r_g}{\lambda}\right)^2\right), \qquad (9)$$

where $H_n(\xi)$ is the Hermite Polynomials,and the

$$n=-1+\left(\frac{2r_g}{\lambda}\right)^2\left[\frac{E}{mc^2}+1\right]+8\left(\frac{r_g}{\lambda}\right)^4 \qquad (10)$$

denote the positive integer number according to definition of the HermitePolynomials.

Then we have the asymptotic behavior of the φ-function

$$\varphi_n(y)\sim\exp(-y-4\alpha^2\sqrt{y})H_n(\sqrt{y}+4\alpha^2), \qquad (11)$$

where $\alpha = r_g/\lambda$.

In papers [19, 20], the authors have done the normalization condition for the wave functions. They numerically studied the motion and energy spectrum of a particle near the blackhole. Results of their numerical studies of behavior of the wave function of a particle are similar to behavior of the wave functions in a problem of the hydrogen atom. Here we shall not investigate a question about normalization of the φ-function, and we shall consider its approach behavior. The authors of articles [19, 20] numerically solved the problem of the Sturm-Liouvilletype. In particular, we are interested in the behavior of a particle near the massive body. Figure 2a illustrates the ground state of a particle bound gravitationally to a central mass. It is necessary to note that the behavior of wave function reminds gravitational analog of a problem of hydrogen atom. The particle may go from the normal state into an excited state of the discrete spectrum. Figure 2b show plots of the 1st (blue solid line and blue dashed line) excited states of the particle. It is necessary to note that the impenetrable-sphere pushes out a particle outside from deeper regions of potential.

In paper [21], the authors have analyzed the sensitive constant α. According to [21], the range of α has values from 0.1 to 0.5. A more detailed analysis of the range of α is described by them. The lowest limit of α is taken to be 0.1. According to [22], for an axion-like particle with a value of α = 0.4, the formed peaks of cloud are close to the domain where dominating radiations come from. The stationary states of wave functions for the region of α are different. Therefore, we consider two of α values. In Fig. 2, we have depicted them. As can be seen in Fig. 2, the wave functions change with increasing α.

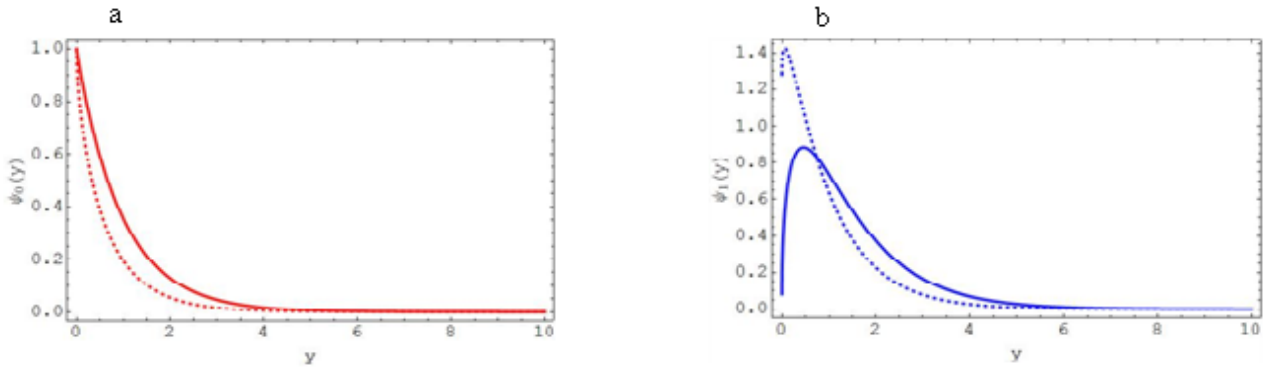


Figure 2. Behavior of wave functions of the ground states (a). According to [21], the lowest limit of α = 0.1 (red solid line). At the increasing α = 0.4 (red dashed line). Behavior of wave functions of the 1st excited states. For a blue solid line α = 0.1, for a blue dashed line α = 0.4 (b).

According to the definition (10), we thus obtain for the energy levels the values

$$E_n=-2mc^2\alpha^2\left[\frac{1}{8\alpha^4}(n+1)-\frac{1}{2\alpha^2}-1\right]. \qquad (12)$$

Thus the distance between two neighboring levels is $-mc^2/4\alpha^2$. Hence we reach the conclusion that, in any part of the spectrum, the levels are equidistant at intervals of $-mc^2/4\alpha^2$.

III. Weak gravitational field

Above, we have considered the behavior of a particle in a Schwarzschild field. Naturally, there is

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an interest in how a particle behaves in a weak gravitational field. To carry out this transition we expand $U(r)$ in powers of r_g/r. Then, neglecting terms of higher order, we find

$$U(r) = -G\frac{mM}{r}. \qquad (13)$$

Eq. (13) can be interpreted also as mutual potential energy of the masses m and M. This equation determines potential energy of a gravitational field of arbitrary mass distribution in the non-relativistic approximation. In Fig. 3, the shapes of the potential energies in Eqs. (2) and (13) are illustrated.

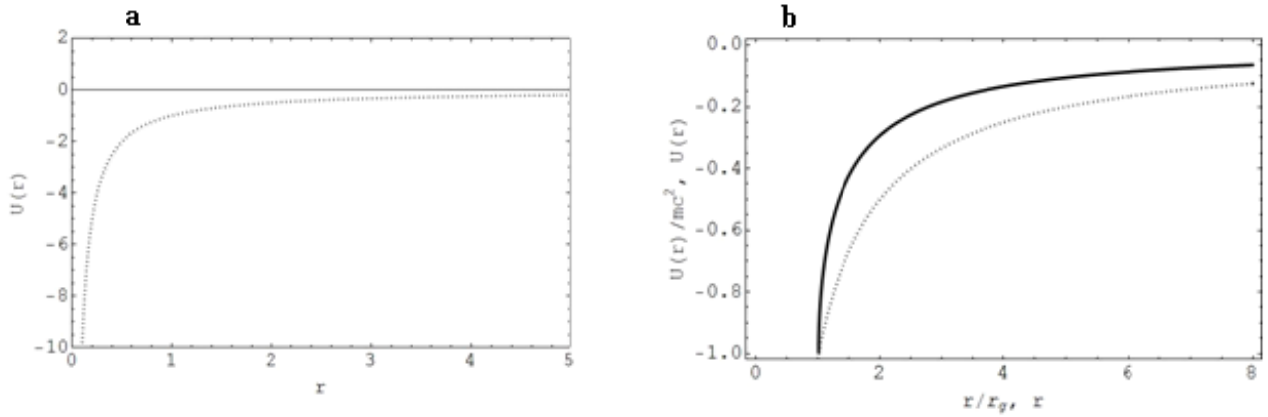


Figure 3. The shape of the potential energy (a) given by Eq. (13). Comparison (b) between relativistic potential energy (solid line) and non-relativistic potential energy (dotted line) of a particle.

Correspondingly, the equation of motion of the particle is

$$\left(1-\frac{r_g}{r}\right)\frac{d^2\varphi}{dr^2}+\frac{1}{2r}\left(4-\frac{3r_g}{r}\right)\frac{d\varphi}{dr}+\frac{2m}{\hbar^2}\left(E+G\frac{mM}{r}\right)\varphi=0. \qquad (14)$$

Thus, repeating the reasoning as in Sec. II, we obtain the dimensionless equation in the form

$$y\frac{d^2\varphi}{dy^2}+\left(y+\frac{1}{2}\right)\frac{d\varphi}{dy}+\left(\frac{r_g}{\lambda}\right)^2\left[\frac{2E}{mc^2}+1-y\right]\varphi=0. \qquad (15)$$

The solution of Eq. (15) has the asymptotic behavior

$$\varphi_n(y) \sim \exp(-\gamma y)\sqrt{y}L_n^{1/2}(\beta y), \qquad (16)$$

where we have introduced the notation, $\gamma=(\beta+1)/2$, $\beta=\sqrt{1+4\alpha^2}$, and $L_n^{1/2}(\beta y)$ is the generalized Laguerre polynomials. According to the definition of generalized Laguerre polynomials, the integer

$$n=\frac{4\alpha^2-1}{4\beta}+\frac{2\alpha^2E}{\beta mc^2}-\frac{3}{4}. \qquad (17)$$

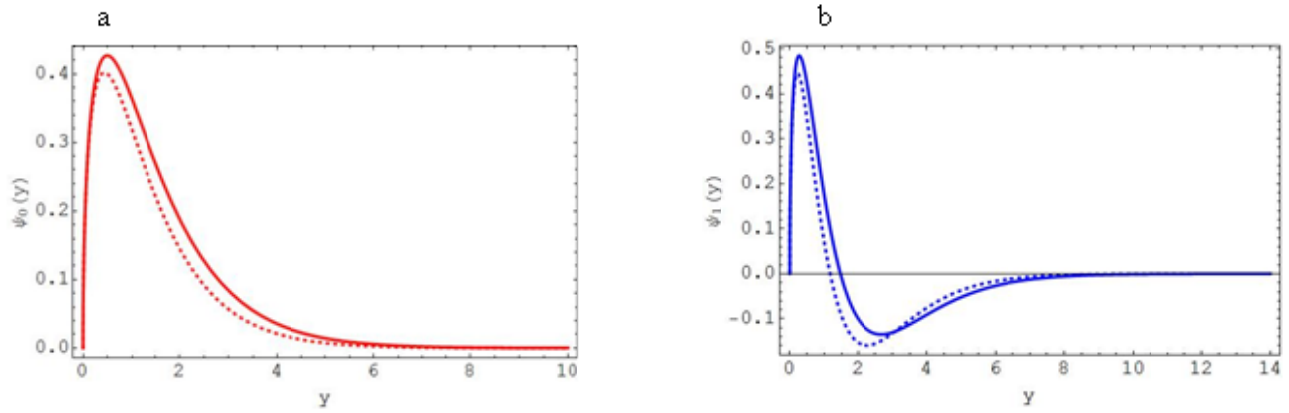


Figure 4. a) Behavior of wave functions of the ground states in a classical case. b) Behavior of wave functions of the 1st excited states in a classical case. All parameters are the same as in Fig. 2.

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In Fig. 4, we show the asymptotic behavior of the wave functions of a particle for the same values of α as in Sec. II. When α increases, the peak of the wave function moves to the left.

According to the definition (17), we thus obtain for the energylevels the values

$$E_n = -\frac{mc^2}{2}\left[\frac{1}{\alpha^2}\left(\frac{1}{4}+\beta(n+\frac{3}{4})\right)-1\right]. \qquad (18)$$

We can see that the levels are equidistant, at intervals $-\beta mc^2 / 2\alpha^2$.

The considered simple model can be partially used for an experimental part of some quantum particles. In Ref. 23 some astrophysical applications were reported (Sgr A*, supermassive black hole in the Galactic center in X-rays). It is possible to note also some applications related to given problem.

IV. Conclusion

Classical problem of motion of a particle in the centrally symmetric gravitational field were studied by different authors [1, 4, 13-15]. For this goal, it is used the Hamilton-Jacobi equation, which is used in general relativity to describe the motion of a particle in the field of force; it is the first-order nonlinear differential equation. We have considered this classical problem from the quantum point of view. In particular, we have interested in the behavior of the particle in the quantum case. In a classical case, the energy of a particle is continuous quantity. In a quantum case, thee nergy of a particle is discrete quantity.

We have limited our consideration to spherically symmetric solutions.Therefore, axially symmetric solutions have been considered for the nonrelativistic quantum case. The given problem considers a semiclassical formulation of gravity, where matter is quantized, but gravity is not. This problem can also be refer to the nonrelativistic Shrödinger-Newton problem [24]. For a Schwarzschild source, the problem has been solved analytically. Behavior of wave function is similar to the behavior of wave function in the problem of hydrogen atom. The energy levels of a particle found out to be equidistant, in contrast to the hydrogen atom.

Further, the given model can be useful for general models that take into account spin in curved space-time. It is necessary to note that such a stationary states formed by the massive body and a particle is nearly related to the analogue of hydrogen atom, with the coupling constant $\alpha = r_g / \lambda$. Especially the constant α is very sensitive to a rotating black hole. Moreover, the given simple model can be developed for a model which takes into account spin of a particle and parameter of rotation of a massive body. Models taking into account dynamic characteristics of particles and massive gravitating bodies can be useful to experimental studies of some astrophysical objects [25-27].

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Спинсиз заррачанинг қора нуқтанинг горизонти якинидаги ҳаракати

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Спинсиз заррачанинг оғир марказий жисм яқинидаги ҳаракати таҳлил қилинган. Заррачанинг тўлқин функцияси ва энергетик спектри олинган. Заррачанинг асосий ва биринчи кўзгалувчан холатлари кўрсатилган. Кўриб чиқилаётган муаммо тортишишнинг ярим классик формуласи билан боғлик булиб, бу ерда материя квантланган, лекин тортишиш эса йук.

Калит сўзлар: Шрёдингер тенглама, Шварцшильд метрикаси, Шварцшильднинг потенциал майдони, энергия сатҳлари

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