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PHASE MIXING AT THE EARLIER EVOLUTION STAGES OF SELF-GRAVITATING SYSTEMS. I. DISK-LIKE SYSTEMS

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Мазкур ишда биз дисксимон ўз гравитацияланувчи системалар фазавий фазосида эволюциянинг дастлабки босқичида муҳим ўрин эгалловчи кучли ностационар стохастик жараёнларни ўргандик. Турли мураккаб характерга эга тасодифий таъсирлар ёрдамида белгиланган фазавий ҳажмнинг ўзини тутишини хаотик таъсирлар модели асосида сонли ҳисоб китоблари амалга оширилди.

В работе изучены сильные нестационарные стохастические процессы, которые имеют место в фазовом пространстве дискообразных самогравитирующих систем на начальной нестационарной стадии их эволюции. Численные расчеты проводили на основе модели хаотического воздействия, согласно которой выделенный фазовый объем испытывает случайные воздействия, имеющие различный и сложный характер.

We study strongly non-stationary stochastic processes which take place in the phase space of disk- like self-gravitating systems at the early stage of their evolution. Numerical calculations were carried out based on the model of chaotic effects according to which the selected phase volume is experienced by random pushes that have diverse and complicated character.

Introduction. As is well known, at the initial nonstationary stage of evolution of a collisionless self-gravitating system the phase mixing takes place always, which tends to bring the system closer to a state that is quasi-stationary in a regular field (see, for example, [1, 2] and references therein). The statistical mechanics of the strongest nonstationary case was first studied by Lynden-Bell [3], who called the corresponding physical process “violent relaxation”. He suggested that the latter can effectively erase all traces of the initial conditions and can lead nonlinear non-stationary system to a certain quasi-equilibrium state. Thus, in [3] a finite coarse grained velocity distribution was predicted, which coincided with the Fermi-Dirac law. After this work, there was a large influx of publications [1] in order to verify the results of Lynden-Bell and his predictions by means of numerical experimental methods. As it turned out, that the prediction about the specified final state can take place only under extremely unsteady initial conditions and, moreover, only within the framework of one-dimensional consideration of the mixing phenomenon. Note that the classification of types of phase mixing depending on the non-stationarity degree of the system was first proposed in [2, 4]. Below we consider only compulsive mixing of the authors of [2, 4] which involves the violent relaxation of Lynden-Bell in the form of a particular case.

During the compulsive phase mixing and especially in the case of limiting type in the form of violent relaxation, any small phase volume of the system cannot evolve independently, but almost completely obeys the general gravitational field, and the latter at an arbitrary point in the phase space can be changed stochastic over time. As a result,

there is obviously a continuous change in the shape of the phase volume which is also having a random nature. It can be noted that the phase volume is subjected to a series of random pushes of varying intensity, playing, for example, rotational or deforming role. Therefore, it is necessary to apply a statistical approach to describe and study the phase evolution of an obviously nonstationary collisionless system.

Phase model of chaotic impacts. According to the model of the author [4], some selected phase volume is subjected to chaotic effects. Due to the strong non-stationarity of the self-gravitating system, the gravitational field changes dramatically not only in time, but also in space with a variable gradient, leading to compulsive mixing in phase space. As much as any point and an arbitrary element of the phase space behave randomly and to describe and study the phase evolution of an obviously non-stationary collisionless system, it is convenient to apply the statistical approach.

The mechanism of statistical stretching can be modeled as follows. Let the impact on the phase element at the moment of time t be described by the matrix M_n, where $n = t/\tau$, τ – the average time interval within which two consecutive punches can be considered random. Then the degree of stretching of the phase element at time t is characterized by the eigenvalues of the matrix

$$R_n = M_n M_{n-1} ... M_1, \tag{1}$$

where M_i are random matrices. Indeed, if there is a sphere in the initial state

$$\vec{q}^{\,2} = 1, \tag{2}$$

where

$$\vec{q} = \vec{q}(x, y, v_x, v_y),$$

then after the first impact we have $\vec{q}_1 = M_1 \vec{q}$. Substituting into (2), we get

$$\vec{q}_1 (M_1 M_1^T)^{-1} \vec{q}_1 = 1, \tag{3}$$

where the symbol T means the matrix transposition. As much as this is an equation of an ellipse, its semi-axes are determined by the eigenvalues of the matrix M_1 and then M_2 ... M_n.

Let M_i = A(k_i) with probability p and $M_i = B(\alpha)$ with probability $1–p$. Any two matrices must satisfy the following conditions:

$$AB \neq BA, \quad A\vec{q} \neq B\vec{q} \neq \vec{q}, \quad \left|\det A\right| = \left|\det B\right| = 1. \tag{4}$$

The sense of the first two conditions is clear, and the latter means the conservation of the phase volume according to the Liouville theorem. As an example, we can take following matrices

$$A(k_1, k_2) = \begin{Bmatrix} k_1 & 0 & 0 & 0 \\ 0 & k_2 & 0 & 0 \\ 0 & 0 & k_1^{-1} & 0 \\ 0 & 0 & 0 & k_2^{-1} \end{Bmatrix} \quad B(\alpha) = \begin{Bmatrix} \cos\alpha & 0 & 0 & -\sin\alpha \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sin\alpha & 0 & 0 & \cos\alpha \end{Bmatrix},$$

where $A(k_i)$ is a deformation matrix, $B(\alpha)$ is a rotation matrix (α is the angle with a random distribution in the interval [0, 2π], k_i is the coefficients of stretching and compression on the main axes).

Let us denote the eigenvalues of the matrix M_n by λ_{ni} (i = 1,2…) and $\lambda_{n1}>\lambda_{n2}>\ldots>\lambda_{n\mu}$, where μ is the dimension of the system. Then the expression

$$\eta_{ni} = (1/n)\ln(\lambda_{ni}/\lambda_{ni+1}) \tag{5}$$

determines the rate of stretching of the volume on time. To determine the statistical characteristics of the process, we need to repeat the calculation of η_{ni} many times and average it by the number of realizations m, hence, we can calculate η_{ni} and its dispersion $\sigma_i(\eta, n)$

$$\sigma_i(\eta,n) = \left[\frac{\sum_{i=1}^{m}\eta_{ni}}{m} - \left(\frac{\sum_{i=1}^{m}\eta_{ni}}{m}\right)^2\right]^{1/2}$$

averaging over m for each n with a value of μ = 4, which corresponds to a disk-shaped system.

Algorithm for calculating the evolution of the phase element. Based on the above method, we wrote a computer program, the algorithm of which is shown below in graphical form (Fig. 1). According to the phase model of the chaotic effects, at the beginning we enter the values $n_{\max}$ (upper limit of the step), $m_{\max}$ (upper limit of realizations number), p (probability of deformation), ep_s (a number close to zero), but later the value of the step varies depending on the values of the dispersion, i.e. with increasing number n, the dispersion $\sigma_i(\eta,n)$ should tend to zero. If the first entered value of the $n_{\max}$ is not sufficient for such approximation, then the step limit will increase and the process will continue until the dispersion is close enough to zero.

One of the main parts of the program is a block where we use the generator of random number that lies in the interval [0,1]. In the beginning we have a circle. Next, the generator of random numbers determines randomly what kind of push will be applied at time t. Thus, the original form begins to change gradually after each push. Using the function that determines the eigenvalues of the matrix, we find λ_{ni}. This operation continues $m_{\max}$ times. At the end of the cycle m, we determine the degree of stretchability of selected phase volume. By averaging each step n over the number of realizations m, the values of dispersion $\sigma_i(\eta,n)$ are determined. But in the process of averaging one requires to accommodate the determination accuracy of λ.

The results of calculating the evolution of a phase element. Based on the phase model of chaotic effects by the method of random matrix and the developed algorithm, we conducted a numerical experiment on random effects on a selected volume.

As noted above, the acting matrices were $A(k_i)$ and $B(\alpha)$, that have deforming and rotational character, respectively. By varying the amount of a particular type of impact, we investigated the behavior of the selected phase volume. As much as we are dealing with a stochastic process, we also tested the central limit theorem of probability theory [5].

Our calculations show that the variance of the products $R = M_1M_2\ldots M_n$ of independent positive random matrices M_i, i = 1, 2, …, n, for large n obey to the law

$$\sigma_i(\eta,n) \sim n^{-1/2}.$$

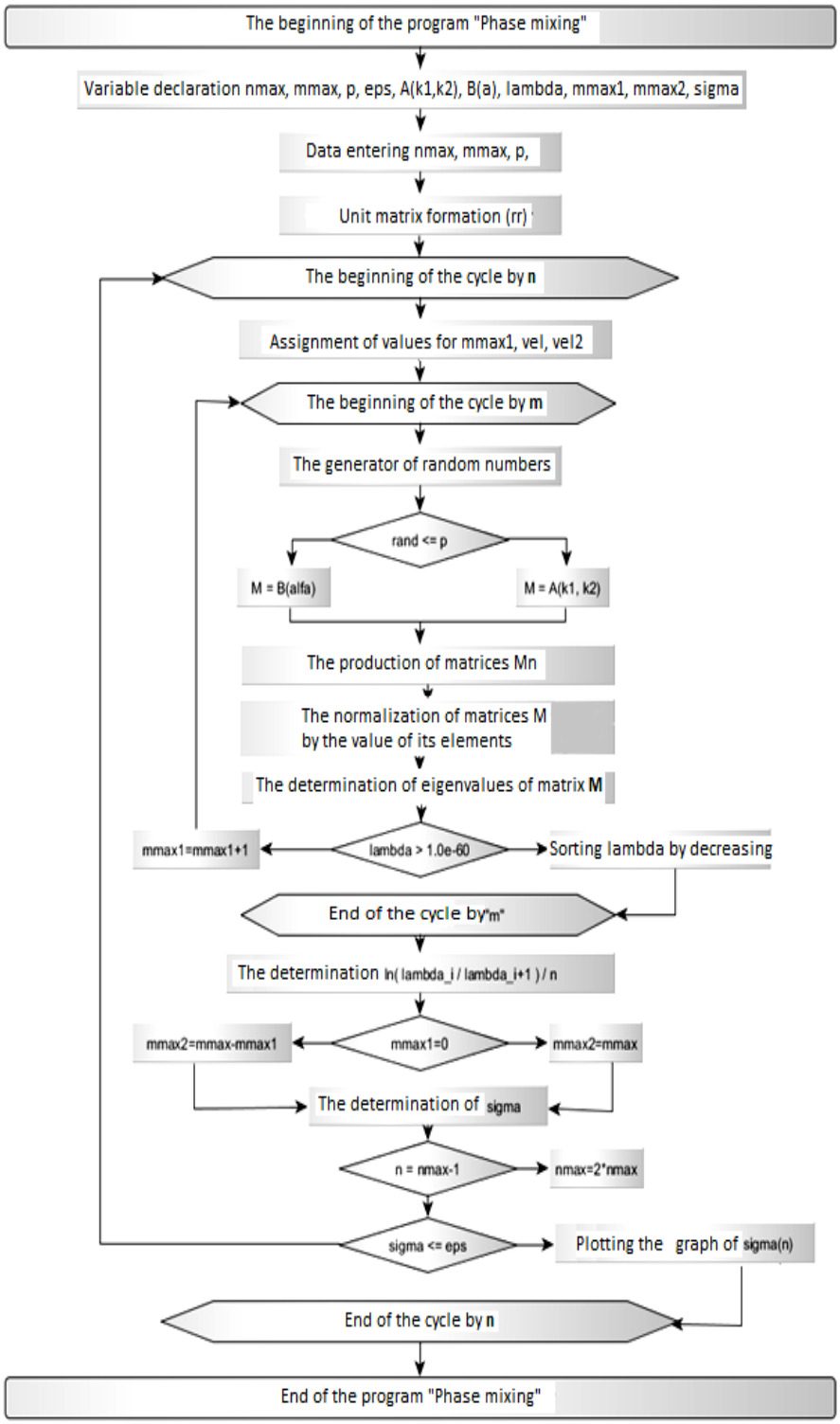


Fig. 1. Algorithm of a computer program for calculating the evolution of a phase element with two types of random matrices.

Fig. 2 shows the results of dispersions and the verification of the above law. From the obtained results it can be seen that the limit of the values of n depending on the value of the probability p varies in the limit $200 \le n \le 1000$, i.e. this limit for this type of random matrix indicates the specific characteristic times of the phase mixing phenomenon which depends on the nature of the effects.

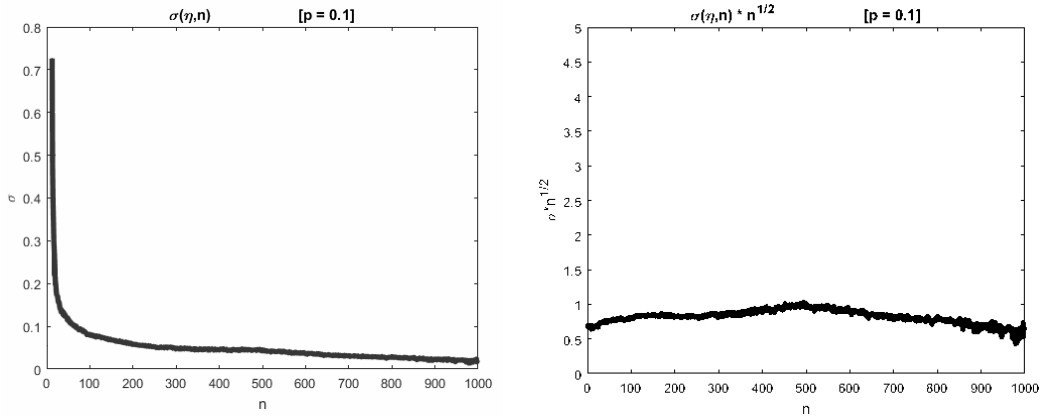


Fig. 2. Dependences of $\sigma_i(\eta, n)$ and $\sigma_i(\eta, n)\sqrt{n^{-1}}$ on step n at $p = 0.1$ and 0.5

Conclusion. Main results of this work are the following. It is simulated compulsive phase mixing process by using production of random matrices. Our modeling takes place in a four-dimensional phase space with two types of matrices. The possibility of choosing suitable matrices describing the effects of different degrees of intensity is shown. Numerical realizations are performed for strongly non-stationary systems by the method of production of random matrices in a four-dimensional phase space. The limit of the values of n changes at $p = 0.1$ five times more than in the case when the contribution of both types of matrix is the same. It is checked the central limit theorem of probability theory which states the law $\sigma_i(\eta, n) \sim n^{-1/2}$.

In the following papers, we will investigate the evolution of the selected phase volume for spherical systems.

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